Geometry and Trigonometry

Map bearings, time difference and nautical miles
True bearing is measured clockwise in degrees from North.
For calculating time difference using longitude:

15^o = 1 hour time difference.

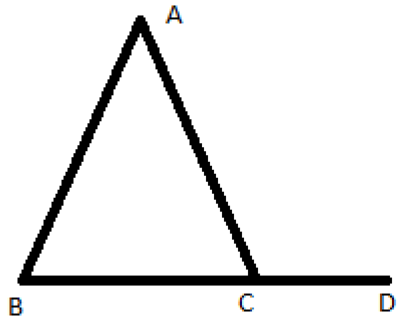
1 nautical mile = 1 minute of latitude = 1 minute of longitude at equator.
At latitude θ^o, 1 minute of longitude is $\cos(\theta)$ nautical miles.

Conventions:
Two dashes across each of a pair of lines indicate they are parallel.
A single dash across each of a pair of line segments indicate that they have equal length.
A small square drawn in the corner of a figure indicates a right angle.

Angle types:

Reflex	$>180^o$, $<360^o$
Acute	$>0^o$, $<90^o$
Obtuse	$>90^o$, $<180^o$
Right	90^o
Straight	180^o

Complementary angles: two angles whose sum is 90^o
Supplementary angles: two angles whose sum is 180^o
Opposite angles: Formed when two lines intersect at a point.
Interior angles: A triangle has 3 interior angles.
Exterior angles: In the diagram below angle ACD is exterior to triangle ABC. It equals the sum of angles CBA and BAC.



2D shapes:
Triangle (3 sides)
Quadrilateral (4 sides)
Trapezium (quadrilateral with 1 pair of opposite sides parallel)
Parallelogram (quadrilateral with 2 pairs of sides parallel)
Rectangle (quadrilateral with each angle 90^o.)
Rhombus (quadrilateral with 4 sides equal)
Square (quadrilateral with each side equal and each angle 90^o.)
Kite (quadrilateral in which 2 pairs of adjacent sides are equal)
Pentagon (5 sides)
Hexagon (6 sides)
Heptagon (7 sides)
Octagon (8 sides)

Notes:

A vertex is a point where two lines meet.
A shape such as a hexagon is regular if all the sides and interior angles are equal.
A right triangle includes a right angle.
Square, rectangle and rhombus are all parallelograms.
Kite and trapezium are not parallelograms.
Square is a rectangle and a rhombus.
Parallelogram is a trapezium.
The diagonals of a parallelogram bisect each other.
The diagonals of a rhombus and a square meet at right angles.
Within a parallelogram adjacent angles add to 180^o and opposite angles are equal.
Triangles: Sum of the angles = 180^o.

- A right triangle includes a right angle.
- An obtuse triangle includes an obtuse angle.
- An isosceles triangle includes two equal angles and has two equal sides.
- An equilateral triangle has three equal sides and three 60^o angles.
- A scalene triangle has three different angles.
- The angles opposite to equal sides of a triangle are equal.
- The incircle of a triangle is the largest circle that can be drawn within a triangle. Its centre is the intersection of the angle bisectors from the vertices.
- The centroid of a triangle is the intersection of the medians. A median is the line from a vertex to the mid-point of the opposite side.

Circles: A chord is a straight line between two points on the perimeter.

- An arc is the part of the perimeter between two points on the perimeter.
- A sector is the area between an arc and the centre.
- A segment is the smaller area between a chord and the circle.
- The tangent to a circle is perpendicular to the radius at the point of contact.
- Tangents to a circle from an external point are equal.
- The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.
- When circles touch, the line through the centres passes through the point of contact.

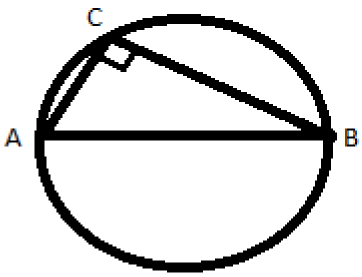
Circumscribed circle of a polygon:

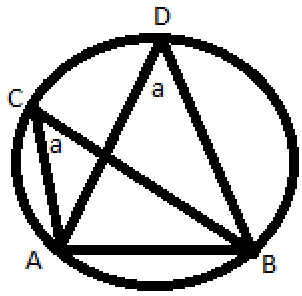
- This passes through all vertices of the polygon.
- The centre is the circumcenter.
- A triangle and rectangle can always be circumscribed.
- A triangle circumcenter is at the intersection of the perpendicular bisectors of the sides.
- The orthocenter of a triangle is the point of intersection of the altitudes. The altitude is the line from a vertex perpendicular to the opposite side. It is not the same as the circumcenter.

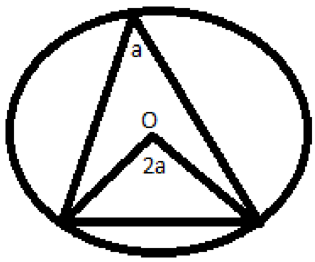
Quadrilaterals: Sum of the internal angles = 360^o.
Cyclic quadrilaterals:

- A quadrilateral is cyclic if a circle can be drawn through its vertices.
- For a cyclic quadrilateral:
  - sum of opposite angles = 180^o.
  - if the opposite angles in a quadrilateral are supplementary then the quadrilateral is cyclic.
  - a diameter subtends a right angle. (see diagram)
  - a single chord subtends a constant angle. (see diagram)

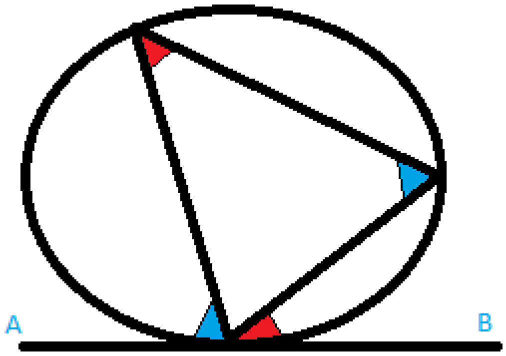
a chord subtends an angle at the centre which is twice the angle subtended at a point on the circle. (see diagram). Note that this is true even if the angle of size 2a shown at right below is reflex.







Alternate segment theorem
In the diagram below AB is a tangent to the circle. The red angles are equal, and the blue angles are equal.



Similar and congruent triangles
Two triangles are similar if: two angles are the same (AA), or three sides are in proportion.
Two triangles are congruent if:

- three sides are the same (SSS),
- or two sides are the same and the included angle is the same (SAS),
- or two angles and the included side are the same (ASA),
- or two angles and any corresponding side pair are the same (AAS),
- or for two right triangles if the hypotenuse and one other side or leg are equal in the other triangle (HL).

Congruent triangles can be made to cover each other exactly using some or all of rotation, translation and reflection. For direct congruence this involves rotation and translation. For opposite congruence a reflection is also needed.
CPCT is short for: corresponding parts of congruent triangles.
In the abbreviation ASA, A means two corresponding angles are the same, S means two corresponding sides are the same. Order is significant.

Isosceles triangle: has two equal sides. The angles opposite the equal sides are equal.

Equilateral triangle: has three equal sides. The angles are all 60°.

Pythagoras theorem $c^2 = a^2 + b^2$; a, b and c are the sides of a right angled triangle.

If a, b and c are integers they are called a Pythagorean triple. Examples are: 3, 4, 5; 5, 12, 13; 7, 24, 25.

Triangle sides: If AB is the longest side of triangle ABC then AB < BC + CA.

Sum of interior angles of polygon with n sides

$S = (n - 2) \times 180^o$

The interior angle of a polygon with n equal sides is: $((n - 2)/n) \times 180^o$

3D shapes or solids:
Triangular prism. (base is a triangle)
Rectangular prism. (i.e. base is rectangle)
Square prism
Square based pyramid.
Triangular pyramid
Rectangular pyramid
Cube

Note: diagonal of a unit cube = √3 units.

Box or cuboid (6 faces, 12 edges, 8 vertices)
Sphere
Cone
Frustum (truncated cone)

Section formula
To find x between x1 and x2 in ratio m:n

$x = (mx_1 + nx_2)/(m + n)$

Net:
A net is pattern that you can cut and fold to make a model of a solid shape.

Theorems:
A statement that requires a proof is called a theorem.
e.g.:
The sum of the angles of a triangle is 180^o.

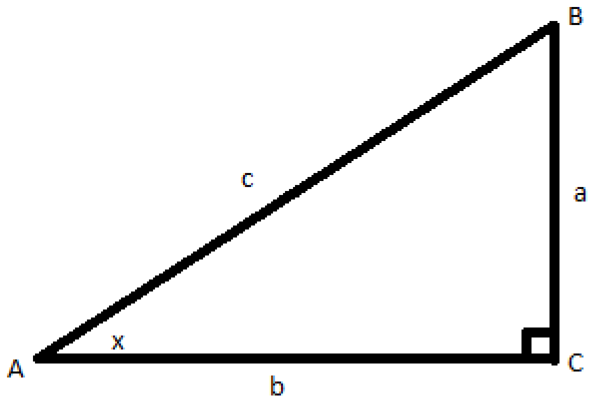
Equidistant points:
A point equidistant from two given points lies on the perpendicular bisector of the line segment joining the two points.
A point equidistant from two intersecting lines lies on the bisectors of the angles joined by the two lines.

Intersecting chords theorem:
Consider two chords AB and CD in a circle radius r that intersect at S distance d from the centre. Then:

$|AS| . |BS| = |CS| . |DS| = r^2 – d^2$

Trigonometry:

Trigonometric identities:



a = opposite side to angle x (also called angle A).
b = adjacent side to angle x.
c = hypotenuse.
sin(x) = a/c
cos(x) = b/c
tan(x) = a/b
sin(-x) = - sin(x)
cos(-x) = cos(x)
tan(-x) = - tan(x)
sec(x) = 1/cos(x)
cosec(x) = 1/sin(x)
cot(x) = 1/tan(x) = $(\tan(x))^{-1}$
cot(x) = tan(90^o - x)
cos(x) = sin(90^o - x)
cosec(x) = sec(90^o - x)
1 + sin(2x) = $(\sin(x) + \cos(x))^2$
$\cos(x)^{-1} = \pi /2 - \sin^{-1}(x)$
$\sin^2 x + \cos^2 x = 1$
$1 + \tan^2 x = \sec^2 x = (\sec(x))^2$
$1 + \cot^2 x = \mathrm{cosec}^2 x$
$\sin^{-1}(a)$ = arcsin(a) = angle whose sine is a.
$\cos^{-1}(a)$ = arccos(a)
$\tan^{-1}(a)$ = arctan(a)

Sine rule:
(for general triangle with sides a, b, c and opposite angles A, B, C):
a/sin(A) = b/sin(B) = c/sin(C)

Cosine rule:
$a^2 = b^2 + c^2 - 2bc \cos(A)$

Sides of some right-angled triangles (in order a, b, c):
3, 4, 5; 5, 12, 13; 7, 24, 25; 8, 15, 17; 9, 40, 41; 12, 35, 37; 20, 21, 29;
1/sqrt(2), 1/sqrt(2), 1; (A, B, C angles 45^o, 45^o, 90^o)
1, sqrt(3), 2; (A, B, C angles 30^o, 60^o, 90^o)

Period and amplitude:
y = A sin(kx) or y = A cos(kx)

Amplitude |A|, period $2\pi/|k|$, range [-A, A].
y = A tan(kx)
period $\pi/|k|$, range R.

General solutions of trigonometric equations:
Rule for positive result in quadrant (start at top right and rotate anti-clockwise):
ASTC: all science teachers count. (meaning all, sine, tangent, cosine)
So: sin(x) is positive in quadrants 1 and 2.
tan(x) is positive in quadrants 1 and 3.
cos(x) is positive in quadrants 1 and 4.

Equation	Solutions (n is integer)
sin x = a;	$x = n\pi + (-1)^n \sin^{-1} a$
cos x = a;	$x = 2n\pi \pm \cos^{-1} a$
tan x = a;	$x = n\pi + \tan^{-1} a$

Domains and ranges:

Function	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
Domain	[-1, 1]	[-1, 1]	R
Range	$[-\pi/2, \pi/2]$	$[0, \pi]$	$(-\pi/2, \pi/2)$

Angle sums and differences
sin(x + y) = sin(x) cos(y) + cos(x) sin(y)
sin(x - y) = sin(x) cos(y) - cos(x) sin(y)
cos(x + y) = cos(x) cos(y) - sin(x) sin(y)
cos(x - y) = cos(x) cos(y) + sin(x) sin(y)
tan(x + y) = (tan(x) + tan(y))/(1- tan(x) tan(y))
tan(x - y) = (tan(x) - tan(y))/(1+ tan(x) tan(y))

sin (2x) = 2 sin (x) cos (x)
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$
$\tan(2x) = (2\tan(x))/(1 - \tan^2(x))$

Let t = tan (x/2). Then:
$\sin(x) = 2t/(1 + t^2)$
$\cos(x) = (1 - t^2)/(1 + t^2)$
$\tan(x) = 2t/(1 - t^2)$

Transformations:

Transforming to linearity:

Replace y versus x plot with:

Logarithmic:	y versus $\log_{10}(x)$	or	$\log_{10}(y)$ versus x
Quadratic:	y versus x^2	or	y^2 versus x
Reciprocal:	y versus 1/x	or	1/y versus x

Circles

$180^o = \pi$ radians

Circumference $2\pi r = \pi D$. r = radius, D = diameter.

Length of an arc with angle x degrees $(\pi r x)/180$
Length of an arc with length x radians $r x$

Circle equation $(x - h)^2 + (y - k)^2 = r^2$
Centre is (h, k), radius is r.

A circle is a closed curve all of whose points lie in a plane and are equidistant from the centre point. The distance from the centre is the radius.

The chord of a circle is a line segment joining two points on the circumference of the circle. The diameter is a chord passing through the centre.

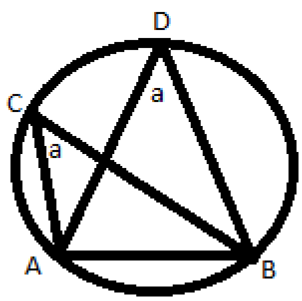
An arc is a part of a circle between two points on the circle.

The region between a chord and either of its corresponding arcs is called a segment (major or minor).

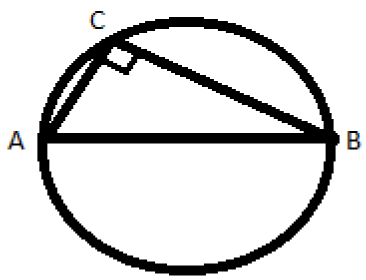
The region between an arc and the two radii joining its ends is called a sector (major or minor).

There is one and only one circle passing through 3 non-collinear points.

The angle subtended by an arc at the centre of a circle is double the angle subtended by it at any point on the circle outside the arc.



The angle subtended by a diameter in a circle is a right angle. This is equivalent to saying that if a circle is drawn through the three points of a right triangle then the hypotenuse is a diameter.



Cyclic quadrilateral:
A quadrilateral is called cyclic if all 4 points lie on a circle.
The sum of opposite angles in a cyclic quadrilateral is 180^o.

Ellipses

An ellipse is a regular oval shape, traced by a point moving in a plane so that the sum of its distances from two other points (the foci) is constant.
Equation: $x^2/a^2 + y^2/b^2 = 1$ (This is centred around the origin and has axes parallel to the coordinate axes.)
A circle is an ellipse for which a = b.
The orbit of a planet is an ellipse. The sun is at a focal point. In the orbit angular momentum and energy are conserved. The orbit around a focal point sweeps out equal areas in equal time.
If a > b, then semi-major axis = a, and semi-minor axis = b.
The area of an ellipse = πab.
The sum of the distances from a point on the ellipse to the foci is 2a.
Ellipticity = e = $\sqrt{(1 - b^2/a^2)}$
Let c = distance of each focus from the centre. Then e = c/a .

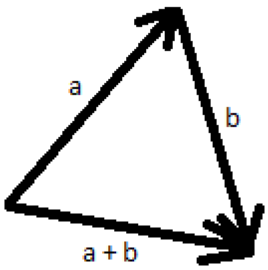
Hyperbolas

Equation: $x^2/a^2 + y^2/b^2 = 1$ (This has asymptotes x = 0, and y = 0.)

Vectors:

A vector is a quantity with magnitude (or length) and direction.
A scalar is a quantity with magnitude but no direction.

Addition of vectors:



Dot product of two vectors with angle θ between them:

$\mathbf{a.b} = a\, b \cos(\theta)$

This is also called the scalar resolute of **a** on **b**.
The dot product is a scalar.
Vectors a and b are parallel if **a.b** = a b
Vectors a and b are perpendicular if **a.b** = 0
If **a** has components (a_1, a_2) and **b** has components (b_1, b_2) then $\mathbf{a.b} = a_1b_1 + a_2b_2$

The modulus of a vector **b** is the scalar length of the vector |**b**| or b.

Projection of vector **a** on **b** = $(\mathbf{a.b})/|\mathbf{b}|^2\ \mathbf{b}$

This is also called the vector resolute or resolute of **a** in the direction **b**.

Linear dependence
A set of vectors are linearly dependent if at least one of them can be written as a linear combination of the others.
One way of proving linear dependence is to make a matrix from the vectors. If the determinant is zero, the vectors are linearly dependent.

Cross product
Cross product of two vectors with angle θ between them:

$\mathbf{a} \times \mathbf{b} = a\, b \sin(\theta)\, \mathbf{n}$

where **n** is a unit vector perpendicular to **a** and **b**.
The direction of **n** is given by the righthand rule: If **a** is in the direction of your second finger, **b** is in the direction of your third finger, then **n** is in the direction of your thumb.

3-D vectors
Let **i**, **j** and **k** be unit vectors in the directions of the x, y and z axes respectively.
These can be used to specify a general vector. E.g.: $\mathbf{r} = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$
$|\mathbf{r}| = r = \sqrt{(x^2 + y^2 + z^2)}$

$d\mathbf{r}/dt = dx/dt\ \mathbf{i} + dy/dt\ \mathbf{j} + dz/dt\ \mathbf{k}$

j is normal to the xz plane. The angle between a vector and this plane = 90^o – angle between **j** and the vector.

Mechanics:
m = mass, **v** = velocity, **a** = acceleration.

Momentum	$\mathbf{p} = m\mathbf{v}$
Force	$\mathbf{F} = m\mathbf{a}$
Friction:	$F <= \mu N$

(here F is the friction force magnitude, N is the normal force magnitude, and μ is the coefficient of friction.)